MATH 2352 Problem Sheet 04

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[Problems] 3.6: 14, 17; 3.7: 7, 10; 3.8: 6, 8; 4.1: 5, 6, 10, 19, 20; 4.2: 25, 34; 4.3: 12.

3.6 - 14. Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

$$\begin{aligned} t^2\, y'' - t(t+2)\, y' + (t+2)\, y &= 6t^3, \quad t>0; \\ y_1(t) &= t, \\ y_2(t) &= t\, e^t. \end{aligned}$$

3.6 - 17. Verify that the given functions y_1 and y_2 satisfy the corresponding homogeneous equation; then find a particular solution of the given nonhomogeneous equation.

$$\begin{aligned} x^2\, y'' - 3x\, y' + 4y &= x^2 \ln x, \quad x>0; \\ y_1(x) &= x^2, \\ y_2(x) &= x^2 \ln x. \end{aligned}$$

3.7 - 7. A mass weighting 3 lb stretches a spring 3 in. If the mass is pushed upward, contracting the spring a distance of 1 in, and then set in motion with a downward velocity of 4 ft/s, and if there is no damping, find the position u of the mass at any time t. Determine the frequency, period, amplitude, and phase of the motion.

3.7 - 10. A mass weighting 16 lb stretches a spring 3 in. The mass is attached to a viscous damper with a damping constant of 2 lb·s/ft. If the mass is set in motion from its equilibrium position with a downward velocity of 6 in/s, find its position u of the mass at any time t. Plot u versus t. Determine when the mass first returns to its equilibrium position. Also find the time τ such that $|u(t)| < 0.01$ in for all $t > \tau$.

3.8 - 6. A mass weighting 5 kg stretches a spring 10 cm. The mass is acted on by an external force of $10\sin(t/2)$ N (newtons) and moves in a medium that imparts a viscous force of 4 N when the speed of the mass is 8 cm/s. If the mass is set in motion from its equilibrium position with an initial velocity of 3 cm/s, formulate the initial value problem describing the motion of the mass.

3.8 - 8. (Continued from 3.8 - 6.)

(a) Find the solution of the initial value problem in Problem 6.

(b) Identify the transient and steady state parts of the solution.

(c) Plot the graph of the steady state solution.

(d) If the given external force is replaced by a force of $2\cos\omega t$ of frequency ω, find the value of ω for which the amplitude of the forced response is maximum.

4.1 - 5. Deternim intervals in which solutions are sure to exist.

$$(x-1)\,y^{(4)}+(x+1)y''+3(\tan x)y \;=\; 0.$$

4.1 - 6. Deternim intervals in which solutions are sure to exist.

$$(x^2-4)y^{(6)}+x^3y'''+9y \;=\; 0.$$

4.1 - 10. Determin whether the given functions are linearly dependent or linearly independent. If they are linearly dependent, find a linear relation among them.

$$\begin{aligned} f_1(t) &= 2t-3, \\ f_2(t) &= t^3+2, \\ f_3(t) &= 2t^2-t, \\ f_4(t) &= t^2+t+1. \end{aligned}$$

4.1 - 19. Let the linear differential operator L be defined by

$$L[y] \;=\; a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_ny,$$

where $a_0a_1,\ldots,a_n$ are real constants.

(a) Find $L[t^n]$.

(b) Find $L[e^{\mathrm{rt}}]$.

(c) Determine four solutions of the equation $y^{(4)}-5y''+4y=0$. Do you think the four solutions form a fundamental set of solutions? Why?

4.1 - 20. In this problem we show how to generalize Theorem 3.2.7 (Abel's theorem) to higher order equations. We first outline the procedure for the third order equation.

$$y'''+p_1(t)y''+p_2(t)y'+p_3(t)y \;=\; 0.$$

Let y_1, y_2 and y_3 be solutions of this equation on an interval I.

(a) If $W=W(y_1,y_2,y_3)$, show that

$$W' \;=\; \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

Hint: The derivative of a 3-by-3 determinant is the sum of three 3-by-3 determinants obtained by differentiating the first, second, and third rows, respectively.

(b) Substitude for y_1''', y_2''' and y_3''' from the differential equation; multiply the first row by p_3, multiply the second row by p_2, and add these to the last row to obtain

$$W' = -p_1(t)\,W.$$

(c) Show that

$$W(y_1, y_2, y_3)(t) = c\exp\left[-\int p_1(t)\,dt\right].$$

It follows that W is either always zero or nowhere zero on I.

(d) Generalize this argument to the nth order equation

$$y_{(n)} + p_1(t)y^{(n-1)} + \cdots + p_n(t)y = 0.$$

with solutions $y_1, ..., y_n$. That is, establish Abel's formula

$$W(y_1, ..., y_n)(t) = c\exp\left[-\int p_1(t)\,dt\right]$$

for this case.

4.2 - 25. Find the general solution of the given differential equation.

$$18y''' + 21y'' + 14y' + 4y = 0.$$

4.2 - 34. Find the solution of the given initial value problem, and plot its graph. How does the solution behave as $t \to \infty$?

$$\begin{aligned} 4y''' + y' + 5y &= 0; \\ y(0) &= 2, \\ y'(0) &= 1, \\ y''(0) &= -1. \end{aligned}$$

4.3 - 12. Find the solution of the given initial value problem. Then plot a graph of the solution.

$$\begin{aligned} y^{(4)} + 2y''' + y'' + 8y' - 12y &= 12\sin t - e^{-t}; \\ y(0) &= 3, \\ y'(0) &= 0, \\ y''(0) &= -1, \\ y'''(0) &= 2. \end{aligned}$$