MATH 2111 Matrix Algebra and Applications Week 13 Tutorial

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Suppose that $\mathbf{w} \in \mathbb{R}^2$ is a vector such that $\mathbf{u} \cdot \mathbf{w} = -1$ and $\mathbf{v} \cdot \mathbf{w} = 3$, find $\mathbf{w}$.

2. Express $\mathbf{u} \cdot \mathbf{v}$ in terms of $||\mathbf{u}+\mathbf{v}||$ and $||\mathbf{u}-\mathbf{v}||$.

3. Let $||\mathbf{u}|| = ||\mathbf{v}|| = ||\mathbf{w}|| = 1$ and $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} = 0$. Find the values of:

$$\text{(i) } (5\mathbf{u}+3\mathbf{v}-2\mathbf{w}) \cdot \mathbf{u} \quad \text{(ii) } (5\mathbf{u}+3\mathbf{v}-2\mathbf{w}) \cdot (4\mathbf{u}-3\mathbf{v}) \quad \text{(iii) } ||5\mathbf{u}+3\mathbf{v}-2\mathbf{w}||^2.$$

4. Let $\mathbf{u}_1 \perp \mathbf{v}_1$ and $\mathbf{u}_2 \perp \mathbf{v}_2$. Do we always have $(\mathbf{u}_1+\mathbf{u}_2) \perp (\mathbf{v}_1+\mathbf{v}_2)$?

5. Let A be an $m \times n$ matrix.

   (i) Show that $\operatorname{Nul}(A^TA) = \operatorname{Nul} A$.

   (ii) Show that $\operatorname{rank}(A^TA) = \operatorname{rank} A$.

   (iii) If $\operatorname{rank} A = n$, show that A^TA is invertible.

   [*When A is $m \times n$ and $\operatorname{rank} A = n$, the matrix $B = (A^TA)^{-1}A^T$ is called the pseudo-inverse of A, satisfying $BA = I_n$.*]

6. Check if the following sets are orthogonal:

$$\text{(i) } \left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\} \quad \text{(ii) } \left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \right\} \quad \text{(iii) } \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}.$$

7. Let A be an $m \times n$ matrix with orthogonal columns. What is special about A^TA?

8. (i) Let W be a subspace of $\mathbf{R}^n$. What is $W \cap W^\perp$?

   (ii) Let $W_1 \subseteq W_2$. What is the relation between $W_1^\perp$ and $W_2^\perp$?

9. Given that $S = \{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is orthogonal and $\mathbf{v} \in \operatorname{Span} S$. Find the number c_2 such that $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3$.

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ -1 \end{bmatrix}, \; \mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ 1 \end{bmatrix}, \; \mathbf{u}_3 = \begin{bmatrix} 3 \\ -1 \\ 3 \\ 4 \end{bmatrix}, \quad \text{(i) } \mathbf{v} = \begin{bmatrix} 10 \\ 1 \\ 16 \\ 7 \end{bmatrix} \quad \text{(ii) } \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}.$$

10. (Optional) Let $V = \mathbb{P}(\mathbb{R})$, the vector space of all real polynomials. Consider the inner product:

$$< p(t), q(t) > = \int_0^1 p(t)q(t)dt.$$

   Let $S = \{1, t^2, t^4, t^6, \ldots\}$. Show that $S^\perp = \{0(t)\}$ (the zero subspace of V).

   [*So in this example $(S^\perp)^\perp = V \neq \operatorname{Span} S$. Therefore, $(S^\perp)^\perp = \operatorname{Span} S$ need not be correct in infinite-dimensional case.*]

11. Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal/orthonormal set in $\mathbb{R}^n$ and let A be an $n \times n$ orthogonal matrix. Is $\{A\mathbf{u}_1, \ldots, A\mathbf{u}_p\}$ again an orthogonal/orthonormal set?

12. Let A, B be $n \times n$ orthogonal matrices. Is (i) $A + B$ (ii) AB always orthogonal?

13. Let A be an orthogonal matrix. What are the possible values of $\det A$?

14. Find all possible 2×2 orthogonal matrices in the form $\begin{bmatrix} \frac{1}{3} & x \\ y & z \end{bmatrix}$.

15. A rigid motion in $\mathbb{R}^n$ is a transformation $M : \mathbb{R}^n \to \mathbb{R}^n$ that preserves distances, namely:

$$||M(\mathbf{u}) - M(\mathbf{v})|| = ||\mathbf{u} - \mathbf{v}|| \quad \text{for every } \mathbf{u}, \mathbf{v} \in \mathbb{R}^n.$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be defined as $T(\mathbf{x}) = M(\mathbf{x}) - M(\mathbf{0})$.

(i) Show that T is also a rigid motion, and preserves lengths: $||T(\mathbf{x})|| = ||\mathbf{x}||$.

(ii) Show that T also preserves inner products: $(T\mathbf{x}) \cdot (T\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

(iii) Now show that T is linear and its standard matrix A is an orthogonal matrix.

[*So, every rigid motion* M *in* $\mathbb{R}^n$ *can be written as* $M(\mathbf{0}) + A\mathbf{x}$*: first an action by an orthogonal matrix* A*, then a translation by adding* $M(\mathbf{0})$.]

16. Given that $\mathcal{B}$ is an orthogonal basis for W. Find $\text{proj}_W \mathbf{v}$ and $\text{dist}(\mathbf{v}, W)$.

$$\mathcal{B} = \{\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}\}; \quad \text{(i) } \mathbf{v} = \begin{bmatrix} 2 \\ -3 \\ 5 \\ -8 \end{bmatrix} \quad \text{(ii) } \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} \quad \text{(iii) } \mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix}.$$

17. Let U, W be two subspaces of $\mathbf{R}^n$ such that $\text{proj}_U \mathbf{v} = \text{proj}_W \mathbf{v}$ for every $\mathbf{v} \in \mathbf{R}^n$. Must we have $U = W$?

18. Let $W_2 \subseteq W_1 \subseteq \mathbb{R}^n$ be a hierarchy of subspaces and $\mathbf{v} \in \mathbb{R}^n$. Set $\mathbf{v}' = \text{proj}_{W_2}(\text{proj}_{W_1} \mathbf{v})$. Is it always true that $\mathbf{v}' = \text{proj}_{W_2} \mathbf{v}$?

19. Let W_1, W_2 be any two subspaces of $\mathbb{R}^n$ and $\mathbf{v} \in \mathbb{R}^n$. Set $\mathbf{v}_1 = \text{proj}_{W_2}(\text{proj}_{W_1} \mathbf{v})$ and $\mathbf{v}_2 = \text{proj}_{W_1}(\text{proj}_{W_2} \mathbf{v})$. Is it always true that $\mathbf{v}_1 = \mathbf{v}_2$?

20. Apply Gram-Schmidt process to the set of vectors $\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ in $\mathbb{R}^n$:

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

Let $W = \text{Span}\{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\}$ and let $\mathbf{u} = (1, 2, a, 2, 1)^T$, where a is a real number. Find $\text{proj}_W \mathbf{u}$. When will the vector $\mathbf{u}$ lie inside W?

21. (Optional) Consider a k-dimensional subspace W of $\mathbb{R}^n$. Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ be an orthogonal basis for W. Extend it to $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{v}_1, \ldots, \mathbf{v}_{n-k}\}$, a basis for $\mathbb{R}^n$. Apply Gram-Schmidt process to change it to $\{\mathbf{w}_1, \ldots, \mathbf{w}_k, \mathbf{u}_1, \ldots, \mathbf{u}_{n-k}\}$, an orthogonal basis for $\mathbb{R}^n$. Show that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-k}\}$ is a basis for $W^\perp$, and hence we have $\dim W + \dim W^\perp = n = \dim \mathbb{R}^n$.

22. Let:

$$A = \begin{bmatrix} 1 & 5 \\ 3 & 1 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 4 \\ -2 \\ -3 \end{bmatrix}.$$

Find a least-squares solution to $A\mathbf{x} = \mathbf{b}$.

23. Find an equation of the straight line that fits each of the following sets of points (x_i, y_i) the best, in the sense of minimizing the square sum of y-distances.

$$\text{(i) } (1,-2),\ (3,4),\ (5,10) \quad \text{(ii) } (1,2),\ (2,5),\ (3,7) \quad \text{(iii) } (2,2),\ (3,3),\ (6,4),\ (9,5).$$

Also write down the error of approximations as square sum of y-distances.
(i.e. error$= \sum_{i=1}^{n}[y_i - (c + mx_i)]^2$.)

24. Find an equation of the quadratic polynomial that fits each of the following sets of points (x_i, y_i) the best, in the sense of minimizing the square sum of y-distances.

$$\text{(i) } (1,4),\ (2,4),\ (3,2),\ (4,-2) \quad \text{(ii) } (-2,1),\ (-1,-2),\ (1,1),\ (2,-1),\ (3,2).$$

Also write down the error of approximations as square sum of y-distances.
(i.e. error$= \sum_{i=1}^{n}[y_i - (c_0 + c_1x_i + c_2x_i^2)]^2$.)

25. Find an orthogonal diagonalization of the symmetric matrix:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix},$$

namely, find an orthogonal matrix P such that P^TAP is a diagonal matrix. Is A positive-definite, negative-definite, or indefinite?

26. Let A be any $m \times n$ matrix. Show that both A^TA and AA^T are positive semi-definite symmetric matrices.

[*Remark: The non-negative square roots of the eigenvalues of A^TA are called the singular values of A. Read §7.4 for the important result on Singular Value Decomposition.*]

Answers for checking:

1. $\begin{bmatrix} -\frac{7}{3} \\ \frac{2}{3} \end{bmatrix}$.
2. $\frac{1}{4}\{||\mathbf{u}+\mathbf{v}||^2 - ||\mathbf{u}-\mathbf{v}||^2\}$.
3. (i) 5 (ii) 11 (iii) 38.
4. No.
5. (i) Note that $||A\mathbf{x}||^2 = \mathbf{x}^TA^TA\mathbf{x}$ (ii) apply rank theorem
6. (i) Yes (ii) Yes (iii) No.
7. diagonal matrix.
8. (i) $\{\mathbf{0}\}$ (ii) $W_2^\perp \subseteq W_1^\perp$.
9. (i) $c_2 = -2$ (ii) $c_2 = 0$.

10. For any $p(t) \in S^{\perp}$, write $p(t) = a_n t^n + \ldots + a_1 t + a_0$. Use the orthogonality of $p(t)$ with $\{1, t^2, t^4, \ldots, t^{2n}\}$ to write down an $(n+1) \times (n+1)$ homogeneous system on $\{a_n, \ldots, a_1, a_0\}$ and claim that the system has unique zero solution.

11. Yes/Yes.

12. (i) No (ii) Yes.

13. ± 1.

14. $\begin{bmatrix} \frac{1}{3} & -\frac{\sqrt{8}}{3} \\ \frac{\sqrt{8}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{8}}{3} \\ \frac{\sqrt{8}}{3} & -\frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & \frac{\sqrt{8}}{3} \\ -\frac{\sqrt{8}}{3} & \frac{1}{3} \end{bmatrix}, \begin{bmatrix} \frac{1}{3} & -\frac{\sqrt{8}}{3} \\ -\frac{\sqrt{8}}{3} & -\frac{1}{3} \end{bmatrix}$.

15. (ii) consider $||T(\mathbf{x}) - T(\mathbf{y})||^2 - ||\mathbf{x} - \mathbf{y}||^2$ and note that $||\mathbf{u} - \mathbf{v}||^2 = ||\mathbf{u}||^2 + ||\mathbf{v}||^2 - 2\,\mathbf{u} \cdot \mathbf{v}$

    (iii) rewrite $||T(\mathbf{x} + \mathbf{y}) - T(\mathbf{x}) - T(\mathbf{y})||^2$, $||T(c\mathbf{x}) - c\,T(\mathbf{x})||^2$ into inner products and use (ii). To show that $A^T A = I_n$, consider $(A\mathbf{e}_i) \cdot (A\mathbf{e}_j) = \mathbf{e}_i \cdot \mathbf{e}_j$.

16. (i) $\begin{bmatrix} 2 \\ -3 \\ 5 \\ -8 \end{bmatrix}, 0$ (ii) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \sqrt{6}$ (iii) $\begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \\ 0 \\ \frac{5}{3} \end{bmatrix}, \sqrt{\frac{2}{3}}$.

17. Yes.

18. Yes.

19. No.

20. $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$; $\operatorname{proj}_W \mathbf{u} = \begin{bmatrix} \frac{2+a}{3} \\ 2 \\ \frac{2+a}{3} \\ 2 \\ \frac{2+a}{3} \end{bmatrix}, a = 1$.

21. First check that $\{\mathbf{u}_1, \ldots, \mathbf{u}_{n-k}\} \subset W^{\perp}$. l.i. is guaranteed. For spanness, let $\mathbf{x} \in W^{\perp}$, and write $\mathbf{x} = c_1\mathbf{w}_1 + \ldots + c_k\mathbf{w}_k + d_1\mathbf{u}_1 + \ldots + d_{n-k}\mathbf{u}_{n-k}$. Use orthogonality to show that $c_1 = \ldots = c_k = 0$.

22. $\begin{bmatrix} \frac{2}{7} \\ \frac{1}{7} \end{bmatrix}$.

23. (i) $y = -5 + 3x$, error $= 0$ (ii) $y = -\frac{1}{3} + \frac{5}{2}x$, error $= \frac{1}{6}$ (iii) $y = \frac{3}{2} + \frac{2}{5}x$, error$= \frac{1}{5}$.

24. (i) $y = 2 + 3x - x^2$, error $= 0$ (ii) $y = -1 - \frac{1}{28}x + \frac{9}{28}x^2$, error $= \frac{47}{7}$.

25. (many choices) $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}$, $P^T A P = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$; indefinite.

26. To determine the sign of eigenvalue λ in $A^T A\mathbf{x} = \lambda\mathbf{x}$, consider $||A\mathbf{x}||^2 = \mathbf{x}^T A^T A\mathbf{x}$.