MATH 2111 Matrix Algebra and Applications Week 12 Tutorial

1. Write down the characteristic equations of the following matrices.

$$\text{(i)} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 3 & 3 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 2 & 5 & 9 \\ 2 & 3 & 11 & 13 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 0 & 0 & -a_0 \\ 1 & 0 & -a_1 \\ 0 & 1 & -a_2 \end{bmatrix} \quad \text{(v)} \begin{bmatrix} 0 & 0 & 0 & -a_0 \\ 1 & 0 & 0 & -a_1 \\ 0 & 1 & 0 & -a_2 \\ 0 & 0 & 1 & -a_3 \end{bmatrix}.$$

2. Find eigenvalues and eigenvectors of the following matrices.

$$\text{(i) } A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 2 & 1 \\ 3 & 1 & 1 \end{bmatrix}.$$

3. Show that A and A^T have the same characteristic equation (and hence the same collection of eigenvalues). Give an example to demonstrate that they might have different collections of eigenvectors.

4. Let λ be an eigenvalue of A. Find an eigenvalue of the following matrices:

$$\text{(i) } A^2 \quad \text{(ii) } A^3 + A^2 \quad \text{(iii) } A^3 + 2I.$$

5. Let $A = R_{\frac{2\pi}{3}}$, the rotation matrix on the plane through $120°$ in anticlockwise direction, fixing the origin. Find the (real) eigenvalues of A and A^3.

6. Find the possible eigenvalues of A if A satisfies:

$$\text{(i) } A^2 = A \quad \text{(ii) } A^2 - 2A + I_n = O \quad \text{(iii) } A \text{ is invertible and } 3A^{-1} = A + 2I \quad \text{(iv) } A^5 = O.$$

7. Let A be an $n \times n$ invertible matrix. Suppose that $\mathbf{v}$ is an eigenvector of A corresponding to eigenvalue λ. Find an eigenvalue and an eigenvector of A^{-1} in terms of λ and $\mathbf{v}$.

8. Let A be an $n \times n$ matrix such that every non-zero vector $\mathbf{x} \in \mathbb{R}^n$ is an eigenvector of A. Show that $A = \lambda I_n$ for some suitable λ.

9. Find a condition on a, b, c such that A is diagonalizable.

$$\text{(i) } A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 2 & c \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

10. Determine whether the following matrices are diagonalizable.

$$\text{(i) } A = \begin{bmatrix} 2 & 3 \\ 0 & 2 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 3 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(iv) } A = \begin{bmatrix} 5 & -4 & 0 & 0 \\ -4 & -1 & 0 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 4 & -1 \end{bmatrix}.$$

11. Let A be an $n \times n$ diagonalizable matrix. Is $I_n + A$ also diagonalizable?

12. Find a pair of diagonalizable matrices A, B such that (i) $A+B$ (ii) AB are <u>not</u> diagonalizable.

13. Diagonalize following matrices:

$$\text{(i) } A = \begin{bmatrix} -4 & -3 & 3 \\ 3 & 2 & -3 \\ -3 & -3 & 2 \end{bmatrix} \quad \text{(ii) } B = \begin{bmatrix} -2 & -3 & 3 \\ 11 & 12 & -3 \\ 5 & 5 & 4 \end{bmatrix}.$$

For (ii), find also a 3×3 matrix C such that $C^2 = B$.

14. Compute A^{100} for each of the following matrices.

$$\text{(i) } A = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

15. Consider the following recurrence relation:

$$a_0 = 1,\ a_1 = 1, \quad a_n = a_{n-1} + 2a_{n-2} \quad \text{when } n \geq 2.$$

(i) Let $\mathbf{x}_n = \begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix}$. Find the matrix A that transforms $\mathbf{x}_n$ to $\mathbf{x}_{n+1}$.

(ii) Find an explicit formula of a_n by using matrix power.

16. A city maintains a constant population of $30,000$. A political study estimated that there are $15,000$ independents, $9,000$ democrats, and $6,000$ liberals. It was also estimated that each year 20% of democrats and 10% of liberals become independents, 20% of independents and 10% of liberals become democrats, 10% independents and 10% of democrats become liberals. Find the political composition of the city in the long run.

17. Let $\mathbf{v}_1$, $\mathbf{v}_2$ be eigenvectors of A, corresponding to distinct eigenvalues $\lambda_1 \neq \lambda_2$ respectively. Will their sum $\mathbf{v}_1 + \mathbf{v}_2$ again be an eigenvector of A?

18. (i) Let P, Q be $n \times n$ matrices. Suppose that $(PQ - I)$ is invertible. Verify that $(QP - I)$ is also invertible with the following inverse:

$$(QP - I)^{-1} = -I + Q(PQ - I)^{-1}P.$$

(ii) Let A, B be $n \times n$ matrices. Show that AB and BA have the same set of eigenvalues (ignoring multiplicities).

Answers for checking:

1. (i) $\lambda^2 - 5\lambda - 2 = 0$ (ii) $(1-\lambda)(2-\lambda)(3-\lambda) = 0$ (iii) $(\lambda^2 - 4\lambda - 1)(\lambda^2 + 1) = 0$
   (iv) $-(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0) = 0$ (v) $\lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0 = 0$.

2. (i) $\lambda = -1 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$, $s \neq 0$; $\lambda = 3 : \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$.

   (ii) $\lambda = 1 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $s \in \mathbb{R}$, $s \neq 0$; $\lambda = 4 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$;

   $\lambda = 6 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u \begin{bmatrix} \frac{8}{5} \\ \frac{5}{2} \\ 1 \end{bmatrix}$, $u \in \mathbb{R}$, $u \neq 0$.

   (iii) $\lambda = 3 : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} \frac{1}{3} \\ 1 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$, $s \neq 0$; $\lambda = \sqrt{2} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -1 - \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$, $t \neq 0$;

   $\lambda = -\sqrt{2} : \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = u \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ -1 + \frac{1}{\sqrt{2}} \\ 1 \end{bmatrix}$, $u \in \mathbb{R}$, $u \neq 0$.

3. (many choices) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

4. (i) λ^2 (ii) $\lambda^3 + \lambda^2$ (iii) $\lambda^3 + 2$.

5. none; 1. (So "λ^3 is an eigenvalue of A^3" $\not\Rightarrow$ "λ is an eigenvalue of A".)

6. (i) $\lambda = 0, 1$ (ii) $\lambda = 1$ (iii) $\lambda = -3, 1$ (iv) $\lambda = 0$.

7. eigenvalue: λ^{-1}, corresponding eigenvector: $\mathbf{v}$.

8. Case $n = 1$ is straightforward; for case $n > 1$, let $\mathbf{x}, \mathbf{y}$ be linearly independent vectors and study the eigenvalues of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{x} + \mathbf{y}$.

9. (i) $a = b = c = 0$ (ii) $a = c = 0$, no condition on b (iii) no condition on a, b, c.

10. (i) No (ii) Yes (iii) No (iv) Yes.

11. Yes, consider $I_n = PI_nP^{-1}$.

12. (e.g.) $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$.

13. (i) $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$

    (ii) $B = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ -1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

    (several choices) $C = \begin{bmatrix} 0 & -1 & 1 \\ 3 & 4 & -1 \\ 1 & 1 & 2 \end{bmatrix}$.

14. (i) $\begin{bmatrix} 1 & 2^{100}-1 \\ 0 & 2^{100} \end{bmatrix}$ (ii) $\begin{bmatrix} 2^{101}-3^{100} & -2^{101}+2\cdot 3^{100} \\ 2^{100}-3^{100} & -2^{100}+2\cdot 3^{100} \end{bmatrix}$

(iii) $\begin{bmatrix} 1 & 2^{100}-1 & 3^{100}-2^{101}+1 \\ 0 & 2^{100} & 2\cdot 3^{100}-2^{101} \\ 0 & 0 & 3^{100} \end{bmatrix}$.

15. (i) $A = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ (ii) $a_n = \frac{1}{3}\{2^{n+1} + (-1)^n\}$.

16. Evenly distributed.

17. Never.

18. (ii) Show that $\det(AB - xI) = 0 \Leftrightarrow \det(BA - xI) = 0$ when $x = 0/x \neq 0$.

[*Remark: Actually, by using block matrix operations:*

$$\begin{bmatrix} I & A \\ O & I \end{bmatrix}^{-1} \begin{bmatrix} AB & O \\ B & O \end{bmatrix} \begin{bmatrix} I & A \\ O & I \end{bmatrix} = \begin{bmatrix} O & O \\ B & BA \end{bmatrix}$$

one can even show that AB and BA are having the same characteristic polynomial.]