1. Find a basis for Nul $A$, where $A$ is given by:

$$\text{(i)} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 1 & -1 \\ 2 & -2 & 1 \\ 3 & -1 & 0 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 1 & 0 & -1 & 1 \\ -1 & 2 & 1 & 2 \\ 2 & 1 & 1 & 0 \end{bmatrix}.$$

2. Find a basis for Row $A$, where $A$ is given by:

$$\text{(i)} \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 1 & 3 & -2 & -1 \\ 2 & -1 & 4 & 3 \\ 3 & -5 & 10 & 7 \\ 4 & 5 & 0 & 1 \\ 5 & 1 & 6 & 6 \end{bmatrix}.$$

3. Find a basis for Col $A$, using only the columns of $A$, where $A$ is given by:

$$\text{(i)} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 3 & 5 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & -1 \\ -1 & -3 & 2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 1 & 2 & 2 & 2 \\ 3 & 2 & 5 & 10 & 7 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 8 & 5 \end{bmatrix}.$$

4. Find a basis for Row $A$ for each matrix $A$ in Q2, using only the rows of $A$.

5. Given that $\mathcal{B}$ is a basis for $\mathbb{R}^3$. Find the $\mathcal{B}$-coordinate vectors of $\mathbf{v}$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ -4 \end{bmatrix} \right\}; \quad \text{(i) } \mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(ii) } \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{(iii) } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}.$$

6. Let $\mathcal{B} = \{1+t, 1-t, t-t^2\}$ be a basis for $\mathbb{P}_2$. Find the vectors (polynomials) $p(t) \in \mathbb{P}_2$ with the following $\mathcal{B}$-coordinate vectors:

$$\text{(i)} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

7. Let $\mathcal{B}$ be a basis of $V$ and let $S$ be a set of vectors in $V$. If the set of $\mathcal{B}$-coordinate vectors of $S$ forms a basis for $\mathbb{R}^n$, is $S$ a basis for $V$?

8. Find the standard matrix of the $\mathcal{B}$-coordinate mapping relative to the following basis for $\mathbb{R}^3$:

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} \right\}.$$

9. Find the dimensions of the following subspaces:

$$\text{(i) } H = \left\{ \begin{bmatrix} s+2t+3u \\ s-t-2u \\ t+u \\ s-3u \end{bmatrix} \mid s,t,u \in \mathbb{R} \right\} \quad \text{(ii) } H = \left\{ \begin{bmatrix} 2s+3t \\ 3s-2t \\ s+t \end{bmatrix} \mid s,t \in \mathbb{R}, s-3t=0 \right\}.$$

10. Find the nullity of each of the following matrices.

$$\text{(i) } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{(iv) } A = \begin{bmatrix} 1 & 3 & 2 & 2 & 4 \\ 2 & 6 & 4 & 3 & 8 \\ 3 & 0 & 6 & 2 & -1 \end{bmatrix}.$$

11. Find the row rank of each of the following matrices.

$$\text{(i) } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \\ 4 & -2 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 2 & -1 & 1 \\ 2 & -1 & 3 & 2 \\ 1 & 7 & -6 & 1 \end{bmatrix}.$$

12. Find the column rank of each of the following matrices.

$$\text{(i) } A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 3 \\ 1 & 0 & -2 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 2 & 3 & 2 & 4 \\ 2 & -1 & 1 & 1 & 3 \\ -1 & 3 & 2 & 1 & 1 \end{bmatrix}.$$

13. Let $A = \begin{bmatrix} -1 & 4 & 6 \\ -3 & 7 & 9 \\ 1 & -2 & -2 \end{bmatrix}$. Determine whether the following vectors are eigenvectors of $A$.

$$\text{(i) } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{(ii) } \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{(iii) } \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \text{(iv) } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \quad \text{(v) } \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

14. For the given matrix $A$ and the given eigenvector $\mathbf{v}$, find the corresponding eigenvalue.

$$\text{(i) } A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$$

$$\text{(iii) } A = \begin{bmatrix} 5 & 9 & 7 \\ 4 & 10 & 7 \\ -8 & -18 & -13 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \quad \text{(iv) } A = \begin{bmatrix} -3 & 6 & 10 \\ -8 & 13 & 20 \\ 4 & -6 & -9 \end{bmatrix}, \ \mathbf{v} = \begin{bmatrix} 5 \\ 0 \\ 2 \end{bmatrix}.$$

15. Consider the same matrices $A$ given in Q14. For the given eigenvalue $\lambda$, find the corresponding collection of eigenvectors.

$$\text{(i) } \lambda = -1 \quad \text{(ii) } \lambda = 1 - \sqrt{3} \quad \text{(iii) } \lambda = 1 \quad \text{(iv) } \lambda = -1.$$

Answers for checking:

1. $\text{(i) } \{ \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \} \quad \text{(ii) } \phi \quad \text{(iii) } \{ \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \\ 1 \end{bmatrix} \} \quad \text{(iv) } \{ \begin{bmatrix} \frac{1}{6} \\ -\frac{3}{2} \\ \frac{7}{6} \\ 1 \end{bmatrix} \}.$

2. $\text{(i) } \{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \} \quad \text{(ii) } \{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \}$

   $\text{(iii) } \{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \} \quad \text{(iv) } \{ \begin{bmatrix} 1 \\ 0 \\ \frac{10}{7} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -\frac{8}{7} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \}.$

3. (i) $\{\begin{bmatrix}1\\2\\3\end{bmatrix},\begin{bmatrix}2\\4\\5\end{bmatrix}\}$ (ii) $\{\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}1\\-1\\2\end{bmatrix}\}$ (iii) $\{\begin{bmatrix}1\\3\\1\\2\end{bmatrix},\begin{bmatrix}1\\2\\1\\1\end{bmatrix},\begin{bmatrix}2\\10\\3\\8\end{bmatrix}\}$.

4. (i) $\{\begin{bmatrix}1\\2\\1\end{bmatrix}\}$ (ii) $\{\begin{bmatrix}1\\2\end{bmatrix},\begin{bmatrix}3\\5\end{bmatrix}\}$ (iii) $\{\begin{bmatrix}-2\\1\\1\end{bmatrix},\begin{bmatrix}1\\-2\\1\end{bmatrix}\}$

   (iv) $\{\begin{bmatrix}1\\3\\-2\\-1\end{bmatrix},\begin{bmatrix}2\\-1\\4\\3\end{bmatrix},\begin{bmatrix}5\\1\\6\\6\end{bmatrix}\}$.

5. (i) $\begin{bmatrix}0\\0\\0\end{bmatrix}$ (ii) $\begin{bmatrix}1\\-1\\-1\end{bmatrix}$ (iii) $\begin{bmatrix}-4\\17\\13\end{bmatrix}$.

6. (i) $0(t)$ (ii) $1+t$ (iii) $3+2t-3t^2$.

7. Yes.

8. $\begin{bmatrix}-1&0&1\\0&2&-3\\1&-1&1\end{bmatrix}$, which is actually $\begin{bmatrix}1&1&2\\3&2&3\\2&1&2\end{bmatrix}^{-1}$.

9. (i) 3 (ii) 1.

10. (i) 3 (ii) 0 (iii) 1 (iv) 2.

11. (i) 0 (ii) 2 (iii) 2.

12. (i) 2 (ii) 3 (iii) 2.

13. (i) Yes (ii) Yes (iii) Yes (iv) No (v) Yes.

14. (i) 3 (ii) $1+\sqrt{3}$ (iii) 0 (iv) 1.

15. (i) $\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}1\\1\end{bmatrix}$, $s\in\mathbb{R}, s\neq 0$ (ii) $\begin{bmatrix}x_1\\x_2\end{bmatrix}=s\begin{bmatrix}-1\\\sqrt{3}\end{bmatrix}$, $s\in\mathbb{R}, s\neq 0$.

    (iii) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}-\frac{9}{4}\\1\\0\end{bmatrix}+t\begin{bmatrix}-\frac{7}{4}\\0\\1\end{bmatrix}$, $s,t\in\mathbb{R}$, not both zeros.

    (iv) $\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=s\begin{bmatrix}1\\2\\-1\end{bmatrix}$, $s\in\mathbb{R}, s\neq 0$.