MATH 2111 Matrix Algebra and Applications Week 9 Tutorial

1. Find $\det A$, $\operatorname{adj} A$ and A^{-1}, where:

$$\text{(i) } A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}, \quad \text{(ii) } A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

2. Let A, B, C, D be $n \times n$ matrices satisfying $CD = DC$. Assuming D is invertible and consider the block matrix M (of size $2n \times 2n$):

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Show that $\det M = \det(AD - BC)$.

Give an example to show that the formula is <u>not</u> necessarily correct if $CD \neq DC$.

3. Let A, B be $n \times n$ invertible matrices. Show that:

$$\text{(i) } \operatorname{adj}(AB) = (\operatorname{adj} B)(\operatorname{adj} A) \qquad \text{(ii) } \operatorname{adj}(\operatorname{adj} A) = (\det A)^{n-2} A.$$

4. Consider the $n \times n$ Cauchy matrix corresponding to two sequences of numbers $\{x_i\}$ and $\{y_j\}$:

$$a_{ij} = \frac{1}{x_i + y_j}, \quad 1 \le i, j \le n. \qquad \text{e.g.} \quad A_2 = \begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} \end{bmatrix}.$$

(i) Write down the $(3, 2)$-cofactor of A_3. What's special about this cofactor?

(ii) Using the inverse formula, find the $(2, 3)$-entry of A_3^{-1}. [Observe the pattern, with emphasis on x_3 and y_2.]

(iii) Recall the determinant formula of Cauchy matrix A_n:

$$\det \left[\frac{1}{x_i + y_j} \right] = \frac{\prod_{1 \le j < i \le n} (x_i - x_j)(y_i - y_j)}{\prod_{1 \le i,j \le n} (x_i + y_j)},$$

identify the terms involving x_ℓ and y_k.

(iv) Note that the (ℓ, k)-cofactor of $\det A_n$ involves an $(n-1) \times (n-1)$ Cauchy matrix with x_ℓ and y_k removed from the original two sequences. Use this (ℓ, k)-cofactor and apply the inverse formula to obtain the (k, ℓ)-entry of A_n^{-1}.

5. Consider the $n \times n$ Hilbert matrix $H_n = (h_{ij})$:

$$h_{ij} = \frac{1}{i + j - 1}, \quad 1 \le i, j \le n. \qquad \text{e.g.} \quad H_3 = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}.$$

[It is a special kind of Cauchy matrix with $x_i = i - 1$, $y_j = j$.]

(i) Using the inverse formula of Cauchy matrix obtained in previous problem, show that the (k, ℓ)-entry of H_n^{-1} is given by:

$$(H_n^{-1})_{k\ell} = \frac{(-1)^{k+\ell}}{k + \ell - 1} \cdot \frac{(n + k - 1)!}{(k-1)!(\ell-1)!(n-\ell)!} \cdot \frac{(n + \ell - 1)!}{(\ell-1)!(k-1)!(n-k)!}.$$

(ii) Show that the (k, ℓ)-entry of H_n^{-1} also takes the following form:

$$(H_n^{-1})_{k\ell} = (-1)^{k+\ell}(k + \ell - 1) \binom{n+k-1}{n-\ell} \binom{n+\ell-1}{n-k} \binom{k+\ell-2}{k-1}^2.$$

[Thus the entries in an inverse Hilbert matrix are all integers.]

6. Let V be the collection of all ordered pairs (x, y) of real numbers. Define the operations $\oplus$ and $\odot$ by the following rules:

$$\begin{aligned}(x_1, y_1) \oplus (x_2, y_2) &:= (x_1 + x_2, y_1 + y_2)\\ k \odot (x, y) &:= (kx, y), \quad \text{where } k \text{ is a real number.}\end{aligned}$$

   Does V form a vector space with the above two operations?

7. Consider the vector space $\mathbb{R}^2$. Let $H = \{\begin{bmatrix} x \\ y \end{bmatrix} : x \geq 0\}$ be the collection of vectors with non-negative x-coordinates. Is H a subspace of $\mathbb{R}^2$?

8. Consider the vector space $\mathbb{R}^3$. Let $H = \{\begin{bmatrix} s + 2t \\ 2s - t \\ -s + t \end{bmatrix} : s, t \in \mathbb{R}\}$. Is H a subspace of $\mathbb{R}^3$?

9. Consider the vector space $M_{n \times n}$ of $n \times n$ matrices, defined using the matrix addition and scalar multiplication. Let K be the subcollection of all $n \times n$ skew-symmetric matrices. Is K a subspace of $M_{n \times n}$?

10. Let H (respectively K) be the collection of all invertible (respectively non-invertible) $n \times n$ matrices. Will H (respectively K) form a subspace of $M_{n \times n}$?

11. Fix a number $b \in \mathbb{R}$. Let H be the collection of all polynomials $p(t)$ with a zero at $t = b$, i.e. $p(b) = 0$. Is H a subspace of $\mathbf{P}$, the vector space of all polynomials? How about the collection K of polynomials $q(t)$ such that $q(b) = 1$?

Answers for checking:

1. (i) $\det A = -2$, $\operatorname{adj} A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$

   (ii) $\det A = -1$, $\operatorname{adj} A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & -1 & 6 \\ 2 & 1 & -5 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$.

2. Consider also the block matrix: $N = \begin{bmatrix} D & O \\ -C & I_n \end{bmatrix}$; (many choices) Take $M = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 5 & 1 & 6 & -1 \\ 1 & 7 & 1 & 8 \end{bmatrix}$.

3. Consider the identity: $A \cdot \operatorname{adj} A = (\det A) I_n$.

4. (i) $(-1)^{3+2} \dfrac{(x_1 - x_2)(y_1 - y_3)}{(x_1 + y_1)(x_1 + y_3)(x_2 + y_1)(x_2 + y_3)}$; determinant of a 2×2 Cauchy matrix corresponding to $\{x_1, x_2\}$ and $\{y_1, y_3\}$ with a sign modification.

   (ii) $(-1)^{3+2} \dfrac{(x_1 + y_2)(x_2 + y_2)(x_3 + y_1)(x_3 + y_2)(x_3 + y_3)}{(x_1 - x_3)(x_2 - x_3)(y_1 - y_2)(y_2 - y_3)}$

   also the same as $(x_3 + y_2) \cdot (-1)^{3+2} \dfrac{(x_1 + y_2)(x_2 + y_2)}{(-1)^{3-1}(x_3 - x_1)(x_3 - x_2)} \dfrac{(x_3 + y_1)(x_3 + y_3)}{(-1)^{2-1}(y_2 - y_1)(y_2 - y_3)}$

   so $(A_3^{-1})_{23} = (x_3 + y_2) \displaystyle\prod_{i=1, i \neq 3}^{3} \frac{x_i + y_2}{x_3 - x_i} \prod_{j=1, j \neq 2}^{3} \frac{x_3 + y_j}{y_2 - y_j}$.

   (iii) In numerator of $\det A_n$: $(-1)^{\ell-1} \displaystyle\prod_{i=1, i \neq \ell}^{n} (x_\ell - x_i)$ and $(-1)^{k-1} \displaystyle\prod_{j=1, j \neq k}^{n} (y_k - y_j)$

   In denominator of $\det A_n$: $\dfrac{1}{x_\ell + y_k} \displaystyle\prod_{i=1}^{n} (x_i + y_k) \prod_{j=1}^{n} (x_\ell + y_j)$

   (iv) $(A_n^{-1})_{k\ell} = (x_\ell + y_k) \displaystyle\prod_{i=1, i \neq \ell}^{n} \frac{x_i + y_k}{x_\ell - x_i} \prod_{j=1, j \neq k}^{n} \frac{x_\ell + y_j}{y_k - y_j}$

5. straightforward verifications

6. No.

7. No.

8. Yes.

9. Yes.

10. No/No.

11. Yes/No.