MATH 2111 Matrix Algebra and Applications Week 8 Tutorial

1. Compute the following determinants:

$$\text{(i)} \begin{vmatrix} 2 & 3 \\ 0 & 4 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} \quad \text{(iv)} \begin{vmatrix} 1 & 2 & 2 \\ 2 & -2 & 1 \\ 2 & 1 & -2 \end{vmatrix} \quad \text{(v)} \begin{vmatrix} 2 & 3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{vmatrix}.$$

2. Let:

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}.$$

Find:

(i) $\det A$ (ii) $\det(-A)$ (iii) $\det(2A)$ (iv) $\det B$ (v) $\det(A+B)$.

3. Use suitable cofactor expansions to compute the following determinants:

$$\text{(i)} \begin{vmatrix} 9 & 0 & 0 & 2 \\ 7 & 3 & 2 & 8 \\ 3 & 0 & 0 & 0 \\ 5 & -3 & 1 & 11 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} 0 & 9 & 0 & 2 \\ 7 & -12 & 3 & -1 \\ 0 & 2 & 0 & 0 \\ 1 & 8 & 1 & 2 \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} 1 & 0 & 5 & 0 & 3 \\ 6 & 8 & 9 & 1 & 2 \\ -1 & 0 & 5 & 11 & 4 \\ 0 & 0 & 3 & 0 & 0 \\ 3 & 0 & 13 & 0 & 2 \end{vmatrix}.$$

4. Find the values of the following determinants if $\det A = 2$:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} : \quad \text{(i)} \begin{vmatrix} a & b & c \\ 2d & 2e & 2f \\ g & h & i \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} g & h & i \\ d & e & f \\ a & b & c \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} a+3g & b+3h & c+3i \\ d & e & f \\ g & h & i \end{vmatrix}.$$

5. If an $n \times n$ matrix contains:

(i) a zero row (ii) two identical rows (iii) two proportional rows,

what should its determinant be?

6. Find a relation between $\det A$, $\det B$, and $\det C$:

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}, \quad B = \begin{bmatrix} u & v & w \\ d & e & f \\ g & h & i \end{bmatrix}, \quad C = \begin{bmatrix} a+u & b+v & c+w \\ d & e & f \\ g & h & i \end{bmatrix}.$$

7. Compute the following determinants:

$$\text{(i)} \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} \quad \text{(ii)} \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 2 & 3 & 4 \\ -1 & -2 & 3 & 4 \\ -1 & -2 & -3 & 4 \end{vmatrix} \quad \text{(iii)} \begin{vmatrix} 1 & 2 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 \\ 0 & 2 & 1 & 2 & 0 \\ 0 & 0 & 2 & 1 & 2 \\ 0 & 0 & 0 & 2 & 1 \end{vmatrix}.$$

8. Let x be a real number. Find $\det A$ in terms of x.

$$\text{(i)}\ A = \begin{bmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ x & x^2 & x^3 & 1 \end{bmatrix} \quad \text{(ii)}\ A = \begin{bmatrix} x & x^2 & x^3 & 1 \\ x^2 & x^3 & 1 & x \\ x^3 & 1 & x & x^2 \\ 1 & x & x^2 & x^3 \end{bmatrix} \quad \text{(iii)}\ A = \begin{bmatrix} 1 & x & x & x \\ x & x & x & x \\ x & x & x^2 & x \\ x & x & x & x^3 \end{bmatrix}.$$

Find x such that A is <u>not</u> invertible.

9. Suppose that A is invertible. What is the relation between $\det A$ and $\det A^{-1}$?

10. Let $\det A = 2$, $\det B = 3$, $\det C = -1$. Find the determinants of the following expressions:

$$\text{(i) } ABAC^T, \quad \text{(ii) } B^T A^{-1} C B^{-1} C^T A, \quad \text{(iii) } A^5 B^{-2} (C^T)^2 A^{-1} B A^{-1} C.$$

11. Let $\{x_1, x_2, \ldots, x_n\}$ and $\{y_1, y_2, \ldots, y_n\}$ be two sequences of numbers such that $x_i + y_j \neq 0$ for all i, j. The $n \times n$ Cauchy matrix $A_n = (a_{ij})$ corresponding to $\{x_i\}$, $\{y_j\}$ is defined by:

$$a_{ij} = \frac{1}{x_i + y_j}, \quad 1 \leq i, j \leq n. \qquad \text{e.g.} \quad A_2 = \begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} \end{bmatrix}.$$

[Note: When $x_i = i - 1$, $y_j = j$, we obtain the $n \times n$ Hilbert matrix.]

The following exercises will lead you to a general formula of $\det A_n$.

(i) Find $\det A_2$.

(ii) By applying row replacement operations $-r_3 + r_1$, $-r_3 + r_2$ to A_3, find the factor k in the following equality:

$$\det A_3 = k \det \begin{bmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \frac{1}{x_1+y_3} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \frac{1}{x_2+y_3} \\ 1 & 1 & 1 \end{bmatrix}.$$

(iii) Continue by column replacement operations $-c_3 + c_1$, $-c_3 + c_2$, find the factor ℓ in the following equality:

$$\det A_3 = \ell \det A_2.$$

Then write down the formula of $\det A_3$.

(iv) Find the factor m in the following equality:

$$\det A_4 = m \cdot \det A_3.$$

Then guess a formula of $\det A_n$.

Answers for checking:

1. (i) 8 (ii) -1 (iii) -1 (iv) 27 (v) -11.
2. (i) 3 (ii) 3 (iii) 12 (iv) 1 (v) 15.
3. (i) 54 (ii) -16 (iii) 1848.
4. (i) 4 (ii) -2 (iii) 2.
5. all 0.
6. $\det C = \det A + \det B$ (note that $C \neq A + B$).
7. (i) 5 (ii) 192 (iii) 33.
8. (i) $(1-x^4)^3$, $x = \pm 1$ (ii) $(1-x^4)^3$, $x = \pm 1$ (iii) $x^3(1-x)^3(1+x)$, $x = 0, \pm 1$.
9. $\det A^{-1} = (\det A)^{-1}$.
10. (i) -12 (ii) 1 (iii) $-\frac{8}{3}$.
11. (i) $\dfrac{(x_2-x_1)(y_2-y_1)}{(x_1+y_1)(x_1+y_2)(x_2+y_1)(x_2+y_2)}$ (ii) $k = \dfrac{(x_3-x_1)(x_3-x_2)}{(x_3+y_1)(x_3+y_2)(x_3+y_3)}$;

    (iii) $\ell = \dfrac{(x_3-x_1)(x_3-x_2)(y_3-y_1)(y_3-y_2)}{(x_3+y_1)(x_3+y_2)(x_3+y_3)(x_1+y_3)(x_2+y_3)}$;

    $$\det A_3 = \frac{(x_3-x_1)(x_3-x_2)(x_2-x_1)(y_3-y_1)(y_3-y_2)(y_2-y_1)}{(x_3+y_1)(x_3+y_2)(x_3+y_3)(x_2+y_1)(x_2+y_2)(x_2+y_3)(x_1+y_1)(x_1+y_2)(x_1+y_3)}.$$

    (iv) $m = \dfrac{(x_4-x_1)(x_4-x_2)(x_4-x_3)(y_4-y_1)(y_4-y_2)(y_4-y_3)}{(x_4+y_1)(x_4+y_2)(x_4+y_3)(x_4+y_4)(x_1+y_4)(x_2+y_4)(x_3+y_4)}$;

    $$\det A_n = \frac{\prod_{1\le j<i\le n}(x_i-x_j)(y_i-y_j)}{\prod_{1\le i,j\le n}(x_i+y_j)}$$