1. Find the inverses (if exist) of the following matrices.

$$\text{(i)} \begin{bmatrix} 2 & 5 \\ 3 & 7 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 & 4 \\ 2 & 3 & 5 \\ 3 & 4 & 6 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & 2 & 4 & -5 \\ -1 & -1 & -7 & 8 \\ 0 & 0 & -2 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

2. Write down the elementary matrix that corresponds to the given ERO.

$$\text{(i)} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} -1 & 2 \\ 3 & -1 \\ 1 & 1 \end{bmatrix} \to \begin{bmatrix} -1 & 2 \\ 3 & -1 \\ 0 & 3 \end{bmatrix}.$$

3. (Standard matrix revisited) Consider a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfies:

$$T \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -4 \\ 16 \\ 24 \end{bmatrix}, \quad T \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -17 \\ 31 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} = \begin{bmatrix} -8 \\ 35 \\ 58 \end{bmatrix}.$$

Let A denote the standard matrix of T.

(i) Rewrite the given conditions into a matrix equation $AB = C$. Find also B^{-1}.

(ii) Find the standard matrix A by using the inverse of the matrix B.

4. Find the product P of elementary matrices of the given sequence of EROs (i.e. $B = PA$):

$$\text{(i) } A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B.$$

[i.e. $r_1 \leftrightarrow r_2$, $-r_1 + r_3$, $\frac{1}{2}r_2$, $r_2 + r_3$, $-3r_2 + r_1$.]

$$\text{(ii) } A = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 1 & 3 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = B.$$

[i.e. $r_1 \leftrightarrow r_3$, $-r_1 + r_2$, $-2r_2 + r_3$, $-2r_2 + r_1$.]

5. Find an invertible matrix P such that PA is in RREF.

$$\text{(i)} \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 3 \\ 1 & 3 & 4 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

6. Let A be an $m \times n$ matrix with linearly independent columns. Suppose that P is an $m \times m$ invertible matrix. Will PA again have linearly independent columns?

7. Let A be an $m \times n$ matrix such that the columns of A span $\mathbb{R}^m$. Suppose that Q is an $n \times n$ invertible matrix. Will the columns of AQ again span $\mathbb{R}^m$?

8. Let A be an $m \times n$ matrix ($m \le n$) with rank $A = m$, and let B be the RREF of A. Suppose that P_1, P_2 are two $m \times m$ invertible matrices such that $P_1 A = B = P_2 A$. What can you say about P_1 and P_2?

Answers for checking:

1. (i) $\begin{bmatrix} -7 & 5 \\ 3 & -2 \end{bmatrix}$ (ii) not exist (iii) $\begin{bmatrix} -1 & -2 & 1 & -8 \\ 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$

2. (i) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

3. (i) $A \begin{bmatrix} 1 & 3 & 2 \\ 2 & -2 & 5 \\ 3 & 2 & 7 \end{bmatrix} = \begin{bmatrix} -4 & -17 & -8 \\ 16 & 31 & 35 \\ 24 & 1 & 58 \end{bmatrix}$, $B^{-1} = \begin{bmatrix} 24 & 17 & -19 \\ -1 & -1 & 1 \\ -10 & -7 & 8 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 5 & -5 \\ 3 & -4 & 7 \\ -5 & 1 & 9 \end{bmatrix}$

4. (i) $P_1 = \begin{bmatrix} -\frac{3}{2} & 1 & 0 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -1 & 1 \end{bmatrix}$ (ii) $P_2 = \begin{bmatrix} 0 & -2 & 3 \\ 0 & 1 & -1 \\ 1 & -2 & 2 \end{bmatrix}$

5. (i) $\begin{bmatrix} \frac{3}{5} & \frac{1}{5} \\ -\frac{2}{5} & \frac{1}{5} \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$ (iii) $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 1 & -2 & 1 \end{bmatrix}$ (iv) $\begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$.

6. Yes.

7. Yes.

8. P_1 must be the same as P_2.