1. Identify the domain/codomain of the matrix transformations given by the following matrices:

$$\text{(i) } A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix} \quad \text{(ii) } A = \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 \\ 1 & -2 & 2 & -2 \end{bmatrix} \quad \text{(iii) } A = \begin{bmatrix} 1 & 0 & 2 & -2 \\ 3 & 1 & 4 & -6 \\ 0 & 2 & -4 & 1 \\ -1 & 3 & 5 & 2 \\ 0 & 3 & 7 & 1 \end{bmatrix}$$

2. Are the linear transformations in Ex. 1 one-to-one? onto?

3. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^3$ satisfying the following conditions:

$$T\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -5 \end{bmatrix}, \quad T\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}.$$

Find the standard matrix A of T. Is T one-to-one? onto?

4. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. What can you say about the integers n, m in each of the following cases?

(i) T is one-to-one (ii) T is onto (iii) T is both one-to-one and onto.

5. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. What can you say about the one-to-one and onto properties of T in each of the following cases?

(i) $n > m$ (ii) $n < m$ (iii) $n = m$.

6. Let:

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ -1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix};$$

$$D = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}, \quad E = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 2 \\ 3 & -1 \\ 4 & 2 \end{bmatrix}.$$

Compute the followings if defined:

(i) $A + B - 2C$, (ii) $AD + E$, (iii) $DA + C$, (iv) $BC - EF$, (v) $FE + AD$.

7. When defined, is the product of lower-triangular matrices again lower-triangular?

8. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.

   (i) Find two non-zero solutions $\mathbf{x}_1, \mathbf{x}_2$ of $A\mathbf{x} = \mathbf{0}$.

   (ii) Let $B = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 \end{bmatrix}$. Compute AB.

   (iii) Let C be any 2×2 matrix and $D = B + C$. Do we always have $AC = AD$?

9. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}$. Compute the following matrix products if defined:

(i) $\mathbf{uv}$ (ii) $\mathbf{vu}$ (iii) $\mathbf{u}^T\mathbf{u}$ (iv) $\mathbf{uu}^T$ (v) $\mathbf{uvu}$ (vi) $\mathbf{vuv}$.

10. Let $E = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and let A be any 3×4 matrix.

(i) Compute EA and describe the result.

(ii) Let $P = I_3 + E$. What is PA then?

Answers for checking:

1. (i) domain: $\mathbb{R}^3$, codomain: $\mathbb{R}^2$ (ii) domain/codomain: $\mathbb{R}^4$ (iii) domain: $\mathbb{R}^4$, codomain: $\mathbb{R}^5$.

2. (i) not one-to-one but onto (ii) both one-to-one and onto (iii) one-to-one but not onto

3. $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \\ -3 & -1 \end{bmatrix}$; one-to-one but not onto.

4. (i) $n \leq m$ (ii) $n \geq m$ (iii) $n = m$.

5. (i) not one-to-one; onto unsure (ii) not onto; one-to-one unsure (ii) both unsure.

6. (i) $\begin{bmatrix} -3 & 0 \\ 1 & -2 \end{bmatrix}$ (ii) $\begin{bmatrix} 4 & 4 & -1 \\ 2 & 5 & 4 \end{bmatrix}$ (iii) not defined (iv) $\begin{bmatrix} -12 & -1 \\ -14 & 4 \end{bmatrix}$ (v) not defined.

7. Yes.

8. (i) (many choices) $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (ii) O (iii) Yes.

9. (i) $\begin{bmatrix} 2 & 3 & 4 \\ 4 & 6 & 8 \\ 6 & 9 & 12 \end{bmatrix}$ (ii) $[20]$ (iii) $[14]$ (iv) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$ (v) $\begin{bmatrix} 20 \\ 40 \\ 60 \end{bmatrix}$ (vi) $\begin{bmatrix} 40 & 60 & 80 \end{bmatrix}$.

10. (i) $EA = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 5a_{11} & 5a_{12} & 5a_{13} & 5a_{14} \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (ii) PA is the same as performing ERO: $5r_1 + r_2$ to A.