MATH 2111 Matrix Algebra and Applications — Week 4 Tutorial

1. Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}.$$

   (i) Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix}$. Can the columns of A span $\mathbb{R}^3$?

   (ii) Let $B = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_4 \end{bmatrix}$. Can the columns of B span $\mathbb{R}^3$?

2. (i) Let A be the augmented matrix of a linear system. If each column of A contains a pivot position, the system will have _____ solution.

   (ii) Let A be a 6×4 augmented matrix of a consistent linear system. What is the maximum number of pivot positions in A?

   (iii) If the augmented matrix of a linear system has a pivot position in each row, is the system always consistent?

3. (i) Let $\mathbf{u}, \mathbf{w} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ and let $\mathbf{y} \in \operatorname{Span}\{\mathbf{u}, \mathbf{w}\}$. Is $\mathbf{y} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

   (ii) Let $\mathbf{u}, \mathbf{w} \in \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Is it always true that $\operatorname{Span}\{\mathbf{u}, \mathbf{w}\} = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$?

   (iii) Let $\operatorname{Span}\{\mathbf{v}\} = \operatorname{Span}\{\mathbf{w}\}$. Can we say that $\mathbf{w} = c\mathbf{v}$ for some number c?

   (iv) Let $\operatorname{Span}\{\mathbf{u}, \mathbf{v}\} = \operatorname{Span}\{\mathbf{u}, \mathbf{w}\}$. Can we say that $\mathbf{w} = c\mathbf{v}$ for some number c?

4. Find a set of basic solutions (if any) for each of the following homogeneous systems:

$$\text{(i)} \begin{cases} x_1 + x_3 + x_4 + 2x_5 = 0 \\ 2x_2 + x_3 + 2x_4 + x_5 = 0 \end{cases} \qquad \text{(ii)} \begin{cases} x_1 + 2x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 4x_2 + 4x_3 - 2x_4 = 0 \\ 3x_1 + 6x_2 + 6x_3 - x_4 = 0 \end{cases}$$

$$\text{(iii)} \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ x_1 + x_2 + 2x_3 = 0 \\ 2x_1 + x_2 + x_3 = 0 \end{cases} \qquad \text{(iv)} \begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 0 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 0 \\ -x_1 - x_2 + x_3 - 3x_5 = 0 \end{cases}$$

5. Express the general solutions of the following non-homogeneous systems in terms of the given particular solutions (cf. Ex. 4).

$$\text{(i)} \begin{cases} x_1 + x_3 + x_4 + 2x_5 = -1 \\ 2x_2 + x_3 + 2x_4 + x_5 = 2 \end{cases}, \quad (x_1, x_2, x_3, x_4, x_5) = (-1, 1, 0, 0, 0)$$

$$\text{(iv)} \begin{cases} x_1 + x_2 + 3x_3 + 4x_4 + 3x_5 = 5 \\ 2x_1 + 2x_2 + 2x_4 + 4x_5 = 4 \\ -x_1 - x_2 + x_3 - 3x_5 = -1 \end{cases}, \quad (x_1, x_2, x_3, x_4, x_5) = (1, 1, 1, 0, 0)$$

6. Check if the following sets of vectors are linearly independent/dependent.

$$\text{(i)} \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \quad \text{(ii)} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \end{bmatrix} \right\} \quad \text{(iii)} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 6 \end{bmatrix} \right\} \quad \text{(iv)} \left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

7. Let $\{\mathbf{x}, \mathbf{y}\}$ be a linearly independent set of vectors in $\mathbb{R}^n$. Set:

$$\mathbf{u} = \mathbf{x} + \mathbf{y}, \quad \mathbf{v} = \mathbf{x} - \mathbf{y}, \quad \mathbf{w} = 2\mathbf{x} + 3\mathbf{y}.$$

   Is (i) $\{\mathbf{x}, \mathbf{y}, \mathbf{u}\}$ (ii) $\{\mathbf{u}, \mathbf{v}\}$ (iii) $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ linearly independent?

8. Let $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ and $T = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$, i.e. S is a subset of T ($S \subset T$). Is:

   (i) T l.i. $\Rightarrow S$ l.i? (ii) T l.d. $\Rightarrow S$ l.d.? (iii) S l.i. $\Rightarrow T$ l.i.? (iv) S l.d. $\Rightarrow T$ l.d?

9. Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be vectors such that $\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ is a linearly independent set.
   (i) Write down the definition for $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ being linearly independent.
   (ii) Transform the vector equation in (i) to a vector equation involving the vectors $\mathbf{u}+\mathbf{v}$, $\mathbf{v}+\mathbf{w}$ and $\mathbf{u}+\mathbf{w}$ instead. What are the conditions obtained by the given assumption: "$\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{u}+\mathbf{w}\}$ *is a linearly independent set*"?
   (iii) Show that $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is also linearly independent.

10. (i) Suppose that the columns of an augmented matrix form a linearly independent set. Is the linear system it representing consistent?
   (ii) Suppose that the columns of an augmented matrix form a linearly dependent set. Is the linear system it representing always consistent?
   (iii) Suppose that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are vectors such that $\{\mathbf{u}, \mathbf{v}\}$, $\{\mathbf{u}, \mathbf{w}\}$, $\{\mathbf{v}, \mathbf{w}\}$ are all linearly independent sets. Must $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ be a linearly independent set?
   (iv) If $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ spans $\mathbb{R}^3$, is S always linearly independent?
   (v) If the three columns of a 3×3 matrix A forms a linearly independent set, is the system $A\mathbf{x} = \mathbf{b}$ always consistent for any $\mathbf{b} \in \mathbb{R}^3$?
   (vi) Find the flaw in the following *false* argument:

   Assume that $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent. Then $c_1\mathbf{u} + c_2\mathbf{v} = \mathbf{0}$ has unique trivial solution $c_1 = c_2 = 0$. So $c_1(\mathbf{u}+\mathbf{v}) + c_2(2\mathbf{u}+2\mathbf{v}) = (c_1+2c_2)\mathbf{u} + (c_1+2c_2)\mathbf{v} = 0\mathbf{u} + 0\mathbf{v} = \mathbf{0}$. Therefore $\{\mathbf{u}+\mathbf{v}, 2\mathbf{u}+2\mathbf{v}\}$ is also linearly independent.

Answers for checking:

1. (i) No (ii) Yes.

2. (i) no (ii) 3 (iii) No.

3. (i) Yes (ii) No (iii) Yes (iv) No.

4. (i) $\begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ (iii) no (iv) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$.

5. (i) $\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -2 \\ -\frac{1}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$, where $s, t, u \in \mathbb{R}$ (iv) $\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$, where $s, t \in \mathbb{R}$.

6. (i) l.i. (ii) l.d. (iii) l.i. (iv) l.d.

7. (i) l.d. (ii) l.i. (iii) l.d.

8. (i) Yes (ii) No (iii) No (iv) Yes.

9. (i) $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{0}$ has unique zero solution $x_1 = x_2 = x_3 = 0$
   (ii) $\frac{1}{2}(x_1 + x_2 - x_3)(\mathbf{u} + \mathbf{v}) + \frac{1}{2}(-x_1 + x_2 + x_3)(\mathbf{v} + \mathbf{w}) + \frac{1}{2}(x_1 - x_2 + x_3)(\mathbf{u} + \mathbf{w}) = \mathbf{0}$,
   $\frac{1}{2}(x_1 + x_2 - x_3) = \frac{1}{2}(-x_1 + x_2 + x_3) = \frac{1}{2}(x_1 - x_2 + x_3) = 0$.

10. (i) No (ii) No (iii) No (iv) Yes (v) Yes.