MATH 2111 Matrix Algebra and Applications Week 3 Tutorial

1. Locate the leading entries for each of the following matrices:

$$\text{(i)} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 \\ 2 & 3 \\ 0 & 0 \\ 4 & 0 \\ 0 & 5 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 0 & 2 & 0 & 1 \\ 1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & -1 & -2 & 5 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 0 & 0 & 0 & -3 & 1 \\ 1 & 2 & 1 & 1 & 2 \\ 0 & 3 & 2 & -1 & 0 \end{bmatrix}$$

2. Identify the matrices in REF:

$$\text{(i)} \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(iv)} \begin{bmatrix} 2 & 0 & 1 & 2 & -5 & 0 \\ 0 & 0 & 1 & 3 & -3 & 4 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

3. Assuming those row echelon matrices in Ex.3 are augmented matrices, solve the corresponding linear systems.

4. Transform the following matrices to RREF:

$$\text{(i)} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \quad \text{(ii)} \begin{bmatrix} 1 & 2 & 1 & 1 \\ 3 & 7 & 6 & 5 \\ -2 & -1 & 7 & 4 \end{bmatrix} \quad \text{(iii)} \begin{bmatrix} 2 & 4 & 2 & -2 & 5 & 1 \\ 3 & 6 & 2 & 2 & 0 & 4 \\ 4 & 8 & 2 & 6 & -5 & 7 \end{bmatrix}$$

5. Locate the pivot columns of matrices in Ex.4, and compute also their ranks.

6. Let A be a 2×2 matrix consisting of 0 and 1 only. If A is in RREF, find the number of different forms of A.

7. Consider a linear system with p variables and q equations.

   (a) What are the sizes of the coefficient and augmented matrices?

   (b) If $p > q$, is the system always consistent?

   (c) If $p < q$, is the system always inconsistent?

   (d) If $p = q$, will the system always have a unique solution?

   (e) If $p \neq q$, will the system (when consistent) always have many solutions?

8. Find a suitable b such that the following system is <u>in</u>consistent.

$$\begin{cases} x_1 - x_2 + 2x_3 - x_4 = 1 \\ 2x_1 - x_2 - x_3 + bx_4 = 1 \\ x_1 - 3x_3 - x_4 = -1 \end{cases}$$

9. Find conditions on the b_i's such that the following systems are consistent:

$$\text{(i)} \begin{cases} x_1 + x_2 = b_1 \\ x_2 + x_3 = b_2 \\ x_1 + x_3 = b_3 \end{cases} \quad \text{(ii)} \begin{cases} x_1 + 2x_2 + 2x_4 = b_1 \\ 2x_1 + x_2 - x_3 + 3x_4 = b_2 \\ 3x_2 + x_3 + x_4 = b_3 \end{cases} \quad \text{(iii)} \begin{cases} x_1 - x_3 + x_4 = b_1 \\ x_1 + x_2 + 2x_3 = b_2 \\ 2x_1 + x_2 + x_3 + x_4 = b_3 \\ x_2 + 3x_3 - x_4 = b_4 \end{cases}$$

10. Let A denote the coefficient matrix of a consistent system $A\mathbf{x} = \mathbf{b}$. Determine the number of free variables (if exist) for the following cases.

    (i) A is 3×4, $\operatorname{rank} A = 2$.

    (ii) A is 4×3, $\operatorname{rank} A = 3$.

    (iii) A is 7×5, $\operatorname{rank} A = 4$.

    (iv) A is 7×7, $\operatorname{rank} A = 7$.

11. Let A denote the coefficient matrix of a linear system. Find $\operatorname{rank} A$ in the following cases.

    (i) The system has 20 equations, 30 variables, 10 free variables.

    (ii) The system has 14 equations, 13 basic variables, 10 free variables.

    (iii) A is 5×7, solution set of $A\mathbf{x} = \mathbf{0}$ has 3 independent free parameters.

    (iv) A is 2×3, and $A\mathbf{x} = \mathbf{b}$ is always consistent for every $\mathbf{b}$.

12. Consider the following column vectors in $\mathbb{R}^2$:

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

    Is $\mathbf{u}$ a linear combination of (i) $\{\mathbf{v}_1, \mathbf{v}_2\}$? (ii) $\{\mathbf{v}_1, \mathbf{v}_3\}$?

13. Consider the following column vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

    (i) Is $\mathbf{u}$ a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2\}$? $\{\mathbf{v}_2, \mathbf{v}_3\}$? $\{\mathbf{v}_1, \mathbf{v}_3\}$?

    (ii) Is $\mathbf{u}$ a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$?

14. Let $\mathbf{v}_1, \mathbf{v}_2$ be vectors in $\mathbb{R}^n$. Will $\operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ contain the zero vector $\mathbf{0}$ of $\mathbb{R}^n$?

15. Let $\mathbf{u}$ be a vector in $\mathbb{R}^n$. Is it always true that:

    (i) $\operatorname{Span}\{\mathbf{u}, \mathbf{0}\} = \operatorname{Span}\{\mathbf{u}\}$? (ii) $\operatorname{Span}\{\mathbf{u}\} = \operatorname{Span}\{\mathbf{u}, 2\mathbf{u}\}$?

Answers for checking:

1. (i) $\begin{bmatrix} \boxed{1} \\ 0 \\ \boxed{1} \end{bmatrix}$ (ii) $\begin{bmatrix} \boxed{1} & 2 \\ \boxed{2} & 3 \\ 0 & 0 \\ \boxed{4} & 0 \\ 0 & \boxed{5} \end{bmatrix}$ (iii) $\begin{bmatrix} 0 & \boxed{2} & 0 & 1 \\ \boxed{1} & -2 & 0 & 0 \\ 0 & 0 & 0 & \boxed{4} \\ 0 & \boxed{-1} & -2 & 5 \end{bmatrix}$ (iv) $\begin{bmatrix} 0 & 0 & 0 & \boxed{-3} & 1 \\ \boxed{1} & 2 & 1 & 1 & 2 \\ 0 & \boxed{3} & 2 & -1 & 0 \end{bmatrix}$.

2. (iii) and (iv).

3. (iii) $\begin{cases} x_1 = s \\ x_2 = \frac{1}{2} \\ x_3 = -1 \end{cases}$ where s is free. (iv) $\begin{cases} x_1 = 1 + \frac{1}{2}t \\ x_2 = s \\ x_3 = 13 - 3t \\ x_4 = t \\ x_5 = 3 \end{cases}$ where s, t are free.

4. (i) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 0 & -5 & -3 \\ 0 & 1 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (iii) $\begin{bmatrix} 1 & 2 & 0 & 4 & -5 & 3 \\ 0 & 0 & 1 & -5 & \frac{15}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

5. (i) all 3 columns. rank = 3. (ii) columns 1 and 2. rank = 2. (iii) columns 1 and 3. rank = 2.

6. 5.

7. (a) $q \times p$, $q \times (p+1)$ (b)–(e) No.

8. $b = -2$.

9. (i) no restriction on b_1, b_2, b_3 (ii) $-2b_1 + b_2 + b_3 = 0$ (iii) $\begin{cases} b_1 + b_2 - b_3 = 0 \\ b_1 - b_2 + b_4 = 0 \end{cases}$.

10. (i) 2 (ii) 0 (iii) 1 (iv) 0

11. (i) 20 (ii) 13 (iii) 4 (iv) 2

12. (i) Yes (ii) No.

13. (i) No, No, No (ii) Yes.

14. Yes.

15. (i) Yes (ii) Yes.