MATH 2111: Additional Explanations and Hints to Week 10 Tutorial

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Abstract

This document is provided as a part of supplemental materials for MATH 2111 Matrix Algebra and Applications (2015 autumn). Although it is written in the hope that it will be useful, nothing contained in this document represents the official views or policies of this course. Comments and suggestions are welcomed to be sent to the author (xweiaf@connect.ust.hk).

1 Problem 3

One can prove that A and A^T has the same characteristic equation using the fact that the determinant of matrix $(A - \lambda I)$ is equal to the determinant of its transpose $(A - \lambda I)^T = A^T - \lambda I$. Then we can conclude that A and A^T have the same collection of eigenvalues.

Here we present another proof directly targeting eigenvalues.

Proof. Let λ be an eigenvalue of A, and v is an eigenvector of A corresponding to λ (in particular, v is nonzero). Let $< a, b >$ denote the inner product of two vectors. Let w be an arbitrary vector, then

$$< w, Av >=< w, \lambda v >=< \lambda w, v > . \tag{1}$$

Noticing the fact that for any vectors a, b, we have

$$< a, Ab >=< A^T a, b > . \tag{2}$$

Now we can write Eqn.1 further as

$$< v, \lambda w >=< Av, w >=< v, A^T w > . \tag{3}$$

Therefore, we have

$$< v, (A^T - \lambda I)w >= 0, \tag{4}$$

holds for arbitrary vector w.

This implies that the range of $A^T - \lambda I$ is orthogonal to the line $span(v)$, thus it is not intertible. Ortherwise if it is intertible, then Eqn.4 means that v is orthogonal to any vector in the space, which implies $v = 0$, a contradiction. At last, $A^T - \lambda I$ being not intertible means that λ is an eigenvalue of A^T. □

Remark 1.1. This proof can be generalized to any finite dimensional vector spaces equipped with a nondegenerate bilinear form (not restricted to dot product).

2 Problem 6

When a real function f has the Taylor expansion

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + \dots \tag{5}$$

then a matrix function is defined by substituting x by a matrix A: the powers become matrix powers, the additions become matrix sums and the multiplications become scaling operations. Then $f(A)$ is another matrix when the series converges. And if λ is an eigenvalue of A, $f(\lambda)$ is an eigenvalue of $f(A)$.

Applying this relation to (i), we have $\lambda^2 - \lambda = 0$; (ii): $\lambda^2 - 2\lambda + 1 = 0$; (iii): $3 = \lambda^2 + 2\lambda$; (iv): $\lambda^5 = 0$.

3 Problem 9

For (i), it is noteworthy that A is nilpotent, i.e., $A^4 = 0$. A nilpotent matrix is diagonalizable if and only if it is zero matrix.

Proof. Let $A^n = 0$, and $P^{-1}AP = D$ where D is a diagonal matrix, then

$$0 = A^n = (PDP^{-1})^n = PD^nP^{-1}. \tag{6}$$

Therefore we have $D = 0$, thus $A = 0$. $\square$

4 Problem 18

A remark on how to get the inverse of $(QP - I)$: Given $(PQ - I)$ invertible, then using Taylor expansion of $f(x) = \frac{1}{x-1}$, we have

$$(PQ - I)^{-1} = -I - (PQ) - (PQ)^2 - \dots . \tag{7}$$

Assume that $(QP - I)$ is also intertible, formally we can write its inversion as

$$(QP - I)^{-1} = -I - (QP) - (QP)^2 - \dots . \tag{8}$$

Comparing the two series, we can formally write

$$Q(PQ - I)^{-1}P = [-(PQ) - (PQ)^2 - \dots] = (QP - I)^{-1} + I. \tag{9}$$