MATH 2111: Additional Explanations and Hints to Week 10 Tutorial

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Abstract

This document is provided as a part of supplemental materials for MATH 2111 Matrix Algebra and Applications (2015 autumn). Although it is written in the hope that it will be useful, nothing contained in this document represents the official views or policies of this course. Comments and suggestions are welcomed to be sent to the author (xweiaf@connect.ust.hk).

1 Agile ways to find basis for $Row(A), Col(A)$

Row/column space of a matrix is the vector space spanned by the rows/columns of the matrix, respectively. It is obvious that

$$Row(A) = Col(A^T). \tag{1}$$

To find a basis for $Row(A)/Col(A)$, we can make use of the following lemma:

Lemma 1.1. *$Row(A) = Row(rref(A))$, and $Col(A) = Col_{pivot}(A))$, where $rref(A)$ is the RREF of A, and $Col_{pivot}(A)$ is the vector space spaned by pivot columns (columns containing pivot positions).*

The first equality can be proved by showing that ERO's preserve row space. To prove the second part, we only need to show the dimensions two spaces are equal, since the pivot column space must be a subspace of full column space.

From this lemma , one good news is that since there is an obvious basis for $Row(rref(A))$, i.e., the rows in $rref(A)$ with pivot positions, we have a reliable way to compute it as a basis for $Row(A)$.

And since pivot columns (not rows!) are invariant under ERO's, another good news is that if you want a basis of $Col(A)$ consisting of only A's columns, then the pivot columns are the easiest choice (of course there can be other choices, but obtaining them can be more complicated).

For example, consider the matrix of Problem 3-(ii). First, we perform ERO's on A:

$$\begin{aligned} &\begin{bmatrix} 1 & 3 & 1 \\ 2 & 6 & -1 \\ -1 & -3 & 2 \end{bmatrix} \xrightarrow{-2r_1+r_2, r_1+r_3} \begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 3 \end{bmatrix} \xrightarrow{r_2+r_3} \\ &\begin{bmatrix} 1 & 3 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-r_2/3, -r_2+r_1} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \end{aligned} \tag{2}$$

Therefore, a basis for $Row(A)$ is $\{[1,3,0]^T, [0,0,1]^T\}$; and since the pivot columns are the first and the third one, one basis for $Col(A)$ using only columns of A can be $\{[1,2,-1]^T, [1,-1,2]^T\}$.

Similarly, we can perform ERO's on A^T:

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -3 \\ 1 & -1 & 2 \end{bmatrix} \xrightarrow{-3r_1+r_2, -r_1+r_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & -3 & 3 \end{bmatrix} \xrightarrow{-r_3/3, -2r_3+r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}. \tag{3}$$

Therefore, a basis for $Col(A)$ is $\{[1,0,1]^T, [0,1,-1]^T\}$; in the mean while, we have the pivot columns to be the first and the second ones, that is one basis for $Row(A)$ using only rows of A can be $\{[1,3,1]^T, [2,6,-1]^T\}$.

Remark 1.2. We may also notice the fact that

$$row\_rank(A) = column\_rank(A).$$

Think of A as a coefficient matrix, row rank of A means number of independent equations (constraints), and column rank of A means number of non-free variables. So this is equivalent to say that the number of independent constraints equals to the number of contrained variables.