MATH 2111: Additional Explanations and Hints to Week 9 Tutorial

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Abstract

This document is provided as a part of supplemental materials for MATH 2111 Matrix Algebra and Applications (2015 autumn). Although it is written in the hope that it will be useful, nothing contained in this document represents the official views or policies of this course. Comments and suggestions are welcomed to be sent to the author (xweiaf@connect.ust.hk).

1 Problem 2

Proof. Since D is invertible, formally we can use "block" ERO's to eliminate B, that is,

$$\begin{bmatrix} I & -BD^{-1} \\ 0 & I \end{bmatrix} M = \begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix}. \tag{1}$$

It is easy to see that the determinant of the block ERO matrix is 1; therefore, $det(M)$ equals to the determinant of the block-lower triangular matrix on the right hand side. Now, we can use a formal block row operation to eliminate C in the same way,

$$\begin{bmatrix} A - BD^{-1}C & 0 \\ C & D \end{bmatrix} \begin{bmatrix} I & 0 \\ -CD^{-1} & I \end{bmatrix} = \begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}. \tag{2}$$

It is obvious that the matrix multiplied again has determinant 1. Now, we have

$$\begin{aligned} det(M) &= det\left(\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & D \end{bmatrix}\right) \\ &= det\left(\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}\right) \\ &= det\left(\begin{bmatrix} A - BD^{-1}C & 0 \\ 0 & I \end{bmatrix}\right) det\left(\begin{bmatrix} I & 0 \\ 0 & D \end{bmatrix}\right) \end{aligned} . \tag{3}$$

By cofactor expansions, it is easy seen that for any positive integer m and square matrix P,

$$det\left(\begin{bmatrix} P & 0 \\ 0 & I_m \end{bmatrix}\right) = det\left(\begin{bmatrix} I_m & 0 \\ 0 & P \end{bmatrix}\right) = det(P). \tag{4}$$

Therefore,

$$det(M) = det(A - BD^{-1}C)det(D) = det(AD - BD^{-1}CD). \tag{5}$$

Under the condition that $CD = DC$, the equation above furcher reduces to $det(AD - BC)$. $\square$

When the last condition dose not hold, Equation (5) cannot be further reduced because of existence of counter-examples. For example, let $C = [1, 0; 0, -1]$ be a reflection with respect to x-axis on $\mathbf{R}^2$, and $D = [\cos\gamma, -\sin\gamma; \sin\gamma, \cos\gamma]$ be a rotation with respect to the origin with angle γ in the counter-clockwise direction. To make things simple, we take $\gamma = \pi/2$, and we have

$$CD = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \tag{6}$$

while

$$DC = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \tag{7}$$

Then $det(M) = det(AD - BD^{-1}CD) = det(AD + BC)$. Now if we take $A = [1, 1; 1, 1], B = I_n$, then we have $det(AD - BC) = 1$, wich is not equal to $det(M) = det(AD + BC) = -3$.

2 Problem 4

In general, if A is a Cauthy matrix identified by $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$, then the submatrix A_{lk} of A obtained by erasing l-th row and k-th column of A is also a Cauthy matrix, identified by $(x_1, x_2, \ldots, x_{l-1}, x_{l+1}, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_{k-1}, y_{k+1}, \ldots, y_n)$.

In a previous exercise, we have derived the determinant formula for a general Cauthy matrix. Using that formula, we can directly apply Cramer's rule, and get the (k, l)-th entry of A^{-1} as:

$$\begin{aligned}
(A^{-1})_{k,l} &= (-1)^{k+l} \frac{det(A_{lk})}{det(A)} \\
&= (-1)^{k+l} \frac{\prod_{i<j, i,j \neq l}(x_j - x_i) \prod_{i<j, i,j \neq k}(y_j - y_i)}{\prod_{i \neq l, j \neq k}(x_i + y_j)} \frac{\prod_{1 \leq i,j \leq n}(x_i + y_j)}{\prod_{1 \leq i < j \leq n}(x_j - x_i)(y_j - y_i)} \\
&= (-1)^{k+l} \frac{\prod_{i=l \text{ or } j=k}(x_i + y_j)}{\prod_{i<j, i=l \text{ or } j=l}(x_j - x_i) \prod_{i<j, i=k \text{ or } j=k}(y_j - y_i)} \\
&= (-1)^{k+l} \frac{(x_l + y_k) \prod_{i \neq l}(x_i + y_k) \prod_{j \neq k}(x_l + y_j)}{\prod_{l<j}(x_j - x_l) \prod_{i<l}(x_l - x_i) \prod_{k<j}(y_j - y_k) \prod_{i<k}(y_k - y_i)} \\
&= (-1)^{k+l}(x_l + y_k) \prod_{i \neq l}(x_i + y_k) \prod_{j \neq k}(x_l + y_j) \frac{(-1)^{l-1}}{\prod_{i \neq l}(x_l - x_i)} \frac{(-1)^{l-k}}{\prod_{j \neq k}(y_k - y_j)} \\
&= (x_l + y_k) \prod_{i=1, i \neq l}^{n} \frac{x_i + y_k}{x_l - x_i} \prod_{j=1, j \neq k}^{n} \frac{x_l + y_j}{y_k - y_j}
\end{aligned} \tag{8}$$