MATH 2111: Additional Explanations and Hints to Week 8 Tutorial

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Abstract

This document is provided as a part of supplemental materials for MATH 2111 Matrix Algebra and Applications (2015 autumn). Although it is written in the hope that it will be useful, nothing contained in this document represents the official views or policies of this course. Comments and suggestions are welcomed to be sent to the author (xweiaf@connect.ust.hk).

1 Problem 3-(i)

$$\begin{aligned} det\left(\begin{bmatrix} 9 & 0 & 0 & 2 \\ 7 & 3 & 2 & 8 \\ 3 & 0 & 0 & 0 \\ 5 & -3 & 1 & 11 \end{bmatrix}\right) &= 3 \times det\left(\begin{bmatrix} 0 & 0 & 2 \\ 3 & 2 & 8 \\ -3 & 1 & 11 \end{bmatrix}\right) \\ &= 6 \times det\left(\begin{bmatrix} 3 & 2 \\ -3 & 1 \end{bmatrix}\right) \\ &= 6 \times 9 = 54. \end{aligned} \tag{1}$$

2 Problem 6

The relation is $det(C) = det(A) + det(B)$.

Proof. By cofactor expansion,

$$\begin{aligned}
det(C) =& det\left(\begin{bmatrix} a+u & b+v & c+w \\ d & e & f \\ g & h & i \end{bmatrix}\right) \\
=&(a+u) \times det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - (b+v) \times det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) \\
&+ (c+w) \times det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right) \\
=&\Big\{a \times det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - b \times det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) \\
&+ c \times det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)\Big\} + \\
&\Big\{u \times det\left(\begin{bmatrix} e & f \\ h & i \end{bmatrix}\right) - v \times det\left(\begin{bmatrix} d & f \\ g & i \end{bmatrix}\right) \\
&+ w \times det\left(\begin{bmatrix} d & e \\ g & h \end{bmatrix}\right)\Big\} \\
=&det\left(\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}\right) + det\left(\begin{bmatrix} u & v & w \\ d & e & f \\ g & h & i \end{bmatrix}\right) \\
=&det(A) + det(B).
\end{aligned} \tag{2}$$

□

3 Problem 8-(i)

$$\begin{aligned}
det(A) =& \begin{vmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ x & x^2 & x^3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ 1+x+x^2+x^3 & 1+x+x^2+x^3 & 1+x+x^2+x^3 & 1+x+x^2+x^3 \end{vmatrix} \\
=&(1+x+x^2+x^3) \begin{vmatrix} 1 & x & x^2 & x^3 \\ x^3 & 1 & x & x^2 \\ x^2 & x^3 & 1 & x \\ 1 & 1 & 1 & 1 \end{vmatrix} = (1+x+x^2+x^3) \begin{vmatrix} 0 & x-1 & x^2-1 & x^3-1 \\ 0 & 1-x^3 & x-x^3 & x^2-x^3 \\ 0 & x^3-x^2 & 1-x^2 & x-x^2 \\ 1 & 1 & 1 & 1 \end{vmatrix} \\
=&(1+x+x^2+x^3) \begin{vmatrix} x-1 & x^2-1 & x^3-1 \\ x^3-1 & x^3-x & x^3-x^2 \\ x^3-x^2 & 1-x^2 & x-x^2 \end{vmatrix} = (x^4-1) \begin{vmatrix} 1 & x+1 & x^2+x+1 \\ x^3-1 & x^3-x & x^3-x^2 \\ x^3-x^2 & 1-x^2 & x-x^2 \end{vmatrix} \\
=&(x^4-1) \begin{vmatrix} 1 & x+1 & x^2+x+1 \\ 0 & 1-x^4 & (x+1)(1-x^4) \\ 0 & 1-x^4 & x(1-x^4) \end{vmatrix} = (x^4-1)^3 \begin{vmatrix} 1 & x+1 \\ 1 & x \end{vmatrix} = (1-x^4)^3.
\end{aligned} \tag{3}$$

Therefore, when $x = \pm 1, \pm i$, the matrix A is not invertible.

Remark 3.1. Search "Circulant matrix" to find out more about matrices like A.

4 Problem 11

Let's directly do the derivation for the general case

$$det(A_n) = \begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_n+y_1} & \frac{1}{x_n+y_2} & \cdots & \frac{1}{x_n+y_n} \end{vmatrix}. \tag{4}$$

Substract row n from each of rows 1 to $n-1$, Thus

$$\begin{aligned} a_{ij} \longleftarrow & \frac{1}{x_i+y_j} - \frac{1}{x_n+y_j} \\ =& \frac{x_n - x_i}{(x_i+y_j)(x_n+y_j)} \\ =& \frac{x_n - x_i}{x_n+y_j}\frac{1}{x_i+y_j}. \end{aligned} \tag{5}$$

From the conclusion of Section 2 of this will have no effect on the value of the determinant.

Then we proceed by

1. extracting the factor $\frac{1}{x_n+y_j}$ from column $1 \le j \le n$ (leaving the last row constant), and
2. extracting the factor $x_n - x_i$ from each row $1 \le i \le n-1$.

We have the following:

$$det(A_n) = \left[\prod_{i=1}^{n-1}(x_n - x_i)\right]\left[\prod_{j=1}^{n}\frac{1}{x_n+y_j}\right] \begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & \frac{1}{x_1+y_n} \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & \frac{1}{x_2+y_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}+y_1} & \frac{1}{x_{n-1}+y_2} & \cdots & \frac{1}{x_{n-1}+y_n} \\ 1 & 1 & \cdots & 1 \end{vmatrix} \tag{6}$$

Now, substract column n from each of column 1 to column $n-1$, and row n will be 0 for all but the last column. For the remaining a_{ij}'s, we have:

$$\begin{aligned} a_{ij} \longleftarrow & \frac{1}{x_i+y_j} - \frac{1}{x_i+y_n} \\ =& \frac{y_n - y_j}{(x_i+y_j)(x_i+y_n)} \\ =& \frac{y_n - y_j}{x_i+y_n}\frac{1}{x_i+y_j}. \end{aligned} \tag{7}$$

Again this dose not change the value of determinant, and we proceed by

1. extracting the factor $\frac{1}{x_i+y_n}$ from row $1 \leq i \leq n-1$ (leaving the last column constant), and

2. extracting the factor $y_n - y_j$ from each column $1 \leq j \leq n-1$.

We have

$$det(A_n) = \left[\prod_{i=1}^{n-1}(x_n - x_i)\right]\left[\prod_{j=1}^{n}\frac{1}{x_n + y_j}\right]\left[\prod_{i=1}^{n-1}\frac{1}{x_i + y_n}\right]\left[\prod_{j=1}^{n-1}(y_n - y_j)\right] \begin{vmatrix} \frac{1}{x_1+y_1} & \frac{1}{x_1+y_2} & \cdots & 1 \\ \frac{1}{x_2+y_1} & \frac{1}{x_2+y_2} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{x_{n-1}+y_1} & \frac{1}{x_{n-1}+y_2} & \cdots & 1 \\ 0 & 0 & \cdots & 1 \end{vmatrix} \tag{8}$$

Then via cofactor expansion w.r.t the (n, n)-th entry, we have the following by tiding up the products:

$$det(A_n) = \frac{\prod_{i=1}^{n-1}(x_n - x_i)(y_n - y_i)}{\prod_{1\leq i,j\leq n}(x_n + y_j)(x_i + y_n)} det(A_{n-1}) \tag{9}$$

Here $1 \leq i, j \leq n$ means that (i, j) iterates through each point in $\{1, 2, \ldots, n\}^2$ onece. Finally, using the trivial initial $A_1 = \frac{1}{x_1+y_1}$, we can derive

$$det(A_n) = \frac{\prod_{1\leq i<j\leq n}(x_j - x_i)(y_j - y_i)}{\prod_{1\leq i,j\leq n}(x_i + y_j)} \tag{10}$$