MATH 2111: Additional Explanations and Hints to Week 5 Tutorial

Xiaoyu Wei

October 21, 2015

Abstract

This document is provided as a part of supplemental materials for MATH 2111 Matrix Algebra and Applications (2015 autumn). Although it is written in the hope that it will be useful, nothing contained in this document represents the official views or policies of this course. Comments and suggestions are welcomed to be sent to the author (xweiaf@connect.ust.hk).

1 Problem 6

Proof. Denote $A = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, where $\alpha_i \in \mathbf{R}^m$. Then

$$PA = \begin{bmatrix} P\alpha_1 & P\alpha_2 & \ldots & P\alpha_n \end{bmatrix}. \tag{1}$$

If PA has linearly dependent columns, then there exists $b_i, 1 \leq i \leq n, \sum_{i=1}^n b_i^2 \neq 0$, such that

$$\begin{aligned} 0 &= b_1 P\alpha_1 + b_2 P\alpha_2 + \ldots + b_n P\alpha_n \\ &= Pb_1\alpha_1 + Pb_2\alpha_2 + \ldots + Pb_n\alpha_n \end{aligned}. \tag{2}$$

Multiply both sides with P^{-1}, we have

$$0 = b_1\alpha_1 + b_2\alpha_2 + \ldots + b_n\alpha_n, \tag{3}$$

which contradicts the condition that A has linearly indipendent columns. Therefore, PA must have linearly independent columns. $\square$

2 Problem 7

Proof. Denote $A = [\alpha_1, \alpha_2, \ldots, \alpha_n]$, where $\alpha_i \in \mathbf{R}^m$. Then since the columns of A span $\mathbf{R}^m$, we have $\forall \gamma \in \mathbf{R}^m$,

$$\gamma = x_1\alpha_1 + x_2\alpha_2 + \ldots + x_n\alpha_n = AX = AQQ^{-1}X \tag{4}$$

where $X = [x_1, x_2, \ldots, x_n]'$. Denote $Y = Q^{-1}X$, then it is equivalent to say that $\forall \gamma \in \mathbf{R}^m$,

$$\gamma = (AQ)Y = y_1\beta_1 + y_2\beta_2 + \ldots + y_n\beta_n, \tag{5}$$

where β_i are columns of AQ. Therefore, the columns of AQ span $\mathbf{R}^m$. $\square$

3 Problem 8

We must have $P_1 = P_2$.

Proof. In fact we can prove the statement under weaker conditions, i.e., we don't require B to be the RREF of A. Instead, we only require $P_1 A = P_2 A$, which is equivalent to

$$(P_1 - P_2)A = 0. \tag{6}$$

Denote the i-th row of A as α_i', and the i-th row of $P_1 - P_2$ as $\beta_i' = [b_{i1}, b_{i2}, \ldots, b_{im}]'$. Then $rank(A) = m$ implies that $\{\alpha_k'\}$ are linearly independent.

Equation 6 is a matrix equation, and the i-th row's equation reads

$$\begin{aligned} 0 &= \beta_i' A \\ &= b_{i1}\alpha_1' + b_{i2}\alpha_2' + \ldots + b_{im}\alpha_m'. \end{aligned} \tag{7}$$

Then it immediately follows that $b_{ij} = 0$ for all j from linear independence of $\{\alpha_k'\}$; therefore, $\beta_i' = 0$. The proof above is done for an arbitrary rwo of $P_1 - P_2$, so we have

$$P_1 - P_2 = 0. \tag{8}$$

$\square$